Algebra Universalis, **28** (1991) 324–338

0002-5240/91/030324-15$1.50 + 0.20/0
© 1991 Birkhäuser Verlag, Basel

Elementary equivalent pairs of algebras associated with sets

COLIN NATURMAN AND HENRY ROSE*

Abstract. The elementary equivalence of two full relation algebras, partition lattices or function monoids are shown to be equivalent to the second order equivalence of the cardinalities of the corresponding sets. This is shown to be related to elementary equivalence of permutation groups and ordinals. Infinite function monoids are shown to be ultrauniversal.

Introduction

Given a non-empty set X there are various algebras which are naturally associated with X, for example the symmetric group on X and the partition lattice on X. In particular we can choose X to be a non-zero cardinal. For all cardinals $\lambda > 0$ let Z_λ denote the algebra of some fixed kind (e.g. symmetric groups) associated with λ. Then since the class of all theories in a first order language is a set there must exist pairs of unequal cardinals $\lambda, \kappa > 0$ such that Z_λ and Z_κ are elementary equivalent.

Clearly if λ, $\kappa > 0$ are unequal and λ is finite, then Z_λ and Z_κ are not elementary equivalent. What about the case when λ is infinite? G. Grätzer posed the question of whether all infinite partition lattices are elementary equivalent (see [4], pg 228 IV. 18). J. Ježek [5] showed that all partition lattices on cardinals below $\aleph_\omega$ have distinct first order theories, thus giving a negative answer to Grätzer's question. R. McKenzie [7] found necessary conditions for two infinite symmetric groups to be elementary equivalent. In particular he showed that not all infinite symmetric groups are elementary equivalent. S. Shelah [9] found necessary and sufficient conditions for two symmetric groups to be elementary equivalent.

In this paper we show that for two cardinal $\lambda, \kappa > 0$ the full relation algebras, partition lattices and function monoids on λ, κ are elementary equivalent iff λ and

---

Presented by Walter Taylor.

Received June 27, 1989 and in final form January 26, 1990.

*The work of the second author was supported by a grant from the University of Cape Town Research Committee, and by the Topology Research Group from the University of Cape Town and the South African Council for Scientific and Industrial Research.

κ are second order equivalent. We relate this to the elementary equivalence of the permutation groups. We use our results to generalize Ježek's result to full relation algebras, function monoids and symmetric groups. We suggest some open problems related to our results. □

Notation

λ, κ will denote cardinals while α, β will denote ordinals. X will always denote a non-empty set.

If A, B are structures with appropriate language $\mathscr{L}$ then $A \equiv_{\mathscr{L}} B$ is used to denote that every sentence of $\mathscr{L}$ holding in A holds in B (and hence every sentence holding in B holds in A). If $\mathscr{L}$ is a first order language we will simply write $A \equiv B$. So $A \equiv B$ means that A is elementary equivalent to B. ϕ, ψ will denote formulae, and f will denote a valuation. $B \models_f \phi$ means that the formula ϕ holds in B under the valuation f. $B \models \phi$ means that ϕ holds in B under any valuation.

When dealing with lattices and relation algebras $\prec$ will always denote the covering relation, i.e. $a \prec b$ means $a < b$ and if $a \leq c \leq b$ then either $a = c$ or $b = c$. □

For any non-empty set X consider the following algebras which are naturally associated with X:

1. The full relation algebra on X (see [6] and [8])

$$R_X = \langle \mathscr{R}(X), \cap, \cup, ', \circ, {}^{-1}, \varnothing, X^2, \mathrm{id}_X \rangle$$

where

(a) $\mathscr{R}(X)$ is the set of all binary relations on X i.e. $\mathscr{R}(X) = \mathscr{P}(X^2)$.

(b) $\cup, \cap, ', \varnothing, X^2$ are the usual Boolean operations on $\mathscr{P}(X^2)$.

(c) $\circ$ is binary composition of relations i.e. for

$$\rho, \sigma \in \mathscr{R}(X), \rho \circ \sigma = \{(x, y) \in X^2 : x\rho z \text{ and } z\sigma y \text{ for some } z \in X\}.$$

(d) ${}^{-1}$ is the conversion operation on relations i.e. for

$$\rho \in \mathscr{R}(X), \rho^{-1} = \{(x, y) \in X^2 : y\rho x\}.$$

(e) id_X is the identity relation on X i.e. $\mathrm{id}_X = \{(x, x) : x \in X\}$.

2. The partition lattice on X (see [4])

$$L_X = \langle \mathrm{Part}(X), \cap, + \rangle$$

where
(a) $\mathrm{Part}(X)$ is the set of all partitions (equivalence relations) on X.
(b) $\cap$ is the intersection of partitions (considered as elements of $\mathscr{P}(X^2)$).
(c) $+$ is the join of partitions given by: If $\rho, \tau \in \mathrm{Part}(X)$,

$$\rho + \tau = \bigcap \{\sigma \in \mathrm{Part}(X) : \rho \subseteq \sigma \text{ and } \tau \subseteq \sigma\}.$$

3. The function monoid on X

$$M_X = \langle \mathscr{F}(X), \circ, \mathrm{id}_X \rangle$$

where
(a) $\mathscr{F}(X)$ is the set of all functions with domain and codomain X. (We can consider $\mathscr{F}(X) \subseteq \mathscr{R}(X)$.)
(b) $\circ$, id_X are as in (1).

4. The symmetric group on X

$$S_X = \langle \mathrm{Perm}(X), \circ, {}^{-1}, \mathrm{id}_X \rangle$$

where
(a) $\mathrm{Perm}(X)$ is the set of permutations on X.
(b) $\circ$, ${}^{-1}$, id_X are as in (1). □

Structured sets

Let X be a set. For $n < \omega$ define $\mathscr{P}^n(X)$ inductively as follows: $\mathscr{P}^0(X) = X$. If $\mathscr{P}^n(X)$ has been defined put $\mathscr{P}^{n+1}(X) = \mathscr{P}(\mathscr{P}^n(X))$.

By a *structured set* we mean a pair $\langle X, \mathscr{A} \rangle$ where X is a non-empty set and $\mathscr{A} \subseteq \mathscr{P}^n(\mathscr{W}(X))$ where $n < \omega$ and $\mathscr{W}(X)$ denotes the set of all tuples of elements of X.

We will be concerned with structured sets of the form $\langle \lambda, \mathscr{A} \rangle$ where λ is a non-zero cardinal (hence a non-empty set) and $\mathscr{A} \subseteq \mathscr{R}(\lambda)$.

Consider the following classes of models:

$$\mathbf{R} = \{R_\lambda : \lambda > 0\}$$

$$\mathbf{L} = \{L_\lambda : \lambda \geq 3\}$$

$$\mathbf{M} = \{M_\lambda : \lambda > 0\}$$

$$\mathbf{S} = \{S_\lambda : \lambda > 0\}$$

$$\mathbf{K}^R = \{\langle \lambda, \mathscr{R}(\lambda) \rangle : \lambda > 0\}$$

$$\mathbf{K}^L = \{\langle \lambda, \mathrm{Part}(\lambda) \rangle : \lambda \geq 3\}$$

$$\mathbf{K}^M = \{\langle \lambda, \mathscr{F}(\lambda) \rangle : \lambda > 0\}$$

$$\mathbf{K}^S = \{\langle \lambda, \mathrm{Perm}(\lambda) \rangle : \lambda > 0\}.$$

The **K** classes are classes of structured sets. The reason for considering only $\lambda \geq 3$ in the class **L** and $\mathbf{K}^L$ will be clear later.

The appropriate language for the **K** classes is the following language $\mathscr{L}\mathscr{R}$:

Description of $\mathscr{L}\mathscr{R}$

$\mathscr{L}\mathscr{R}$ has the following alphabet:

Element variables $EV = \{w_i : i < \omega\}$

Relation variables $RV = \{\rho_i : i < \omega\}$

Logical symbols $\{\neg, \wedge, =, (,), \exists\}$.

We will use x, y, z to denote element variables and π, ρ, σ, τ for relation variables.

If $x, y \in EV$ and $\rho \in RV$ then $(x = y)$ and $(x\rho y)$ are atomic formulae. Formulae are built up from atomic formulae as for first order languages, with quantifiers being able to bind both element variables and relation variables.

A structure for $\mathscr{L}\mathscr{R}$ is a structured set of the form $\langle \lambda, \mathscr{A} \rangle$ where $\lambda > 0$ and $\mathscr{A}$ is a non-empty subset of $\mathscr{R}(\lambda)$. A valuation on $\mathscr{L}\mathscr{R}$ for $\langle \lambda, \mathscr{A} \rangle$ assigns element variables to elements of λ and relation variables to relations in $\mathscr{A}$.

Besides the usual abbreviations in logic we also use the following:

For $\rho, \sigma, \tau \in RV$ define:

$\rho \leq \sigma : (\forall x)(\forall y)(x\rho y \Rightarrow x\sigma y)$

$\rho = \sigma : (\forall x)(\forall y)(x\rho y \Leftrightarrow x\sigma y)$

Inter $(\rho, \sigma, \tau) : (\forall x)(\forall y)(x\rho y \Leftrightarrow (x\sigma y \wedge x\tau y))$ i.e. '$\rho = \sigma \cap \tau$'

Union $(\rho, \sigma, \tau) : (\forall x)(\forall y)(x\rho y \Leftrightarrow (x\sigma y \vee x\tau y))$ i.e. '$\rho = \sigma \cup \tau$'

Join $(\rho, \sigma, \tau) : (\forall x)(\forall y)(x\rho y \Leftrightarrow (\forall \pi)((\sigma \leq \pi \wedge \tau \leq \pi) \Rightarrow x\pi y))$

If $\mathscr{A} = \text{Part}(\lambda)$ this says that $\rho = \sigma + \tau$ in the lattice L_λ.

Prod $(\rho, \sigma, \tau) : (\forall x)(\forall y)(x\rho y \Leftrightarrow (\exists z)(x\sigma z \wedge z\tau y))$

i.e. '$\rho = \sigma \circ \tau$'

$\text{Id}(\rho) : (\forall x)(\forall y)(x\rho y \Leftrightarrow x = y)$

i.e. 'ρ is the identity relation'

$\text{Zero}(\rho) : (\forall \sigma)(\rho \leq \sigma)$

i.e. 'ρ is the smallest relation in $\mathscr{A}$'

$\text{One}(\rho) : (\forall \sigma)(\sigma \leq \rho)$

i.e. 'ρ is the largest relation in $\mathscr{A}$'

$\text{Comp}(\rho, \sigma) : (\forall x)(\forall y)(x\rho y \Leftrightarrow \neg x\sigma y)$

i.e. 'ρ is the set-theoretic complement of σ'

$\text{Conv}(\rho, \sigma) : (\forall x)(\forall y)(x\rho y \Leftrightarrow y\sigma x)$

i.e. 'ρ is the converse of σ'.

Note that $\mathscr{L}\mathscr{R}$ is not, strictly speaking, a first order language. However, it is not difficult to see that $\mathscr{L}\mathscr{R}$ is equivalent to a first order language which has only one set of variables (which range over $\lambda \cup \mathscr{A}$) and a single unary predicate (used to distinguish between elements of λ and $\mathscr{A}$). □

Other languages used

Besides $\mathscr{L}\mathscr{R}$ we will also make use of the following languages:

$\mathscr{L}_2$:

This is the full second order language for sets. It contains the usual first order language for sets and has distinct sets of variables for functions and predicates of all finite ranks, all of which may be quantified.

$\mathscr{L}_R$, $\mathscr{L}_L$ and $\mathscr{L}_M$.

These are the usual first order languages for relation algebras, lattices and monoids respectively. (These languages are thus the appropriate languages for the classes R, L and M respectively.)

These languages will have variables $\{x_i : i < \omega\}$ (also denoted by x, y, z), logical symbols $\{\neg, \wedge, =, (,), \exists\}$ and function symbols as follows:

$\mathscr{L}_R : \{\cdot, +, {}', \circ, {}^{-1}, 0, 1, e\}$

$\mathscr{L}_L : \{\cdot, +\}$

$\mathscr{L}_M : \{\circ, e\}$.

We also make implicit use of various other first order languages. □

Interpretations between classes of structures

DEFINITION. Let $\mathbf{A} = \{A_i : i \in I\}$ and $\mathbf{B} = \{B_i : i \in J\}$ be two classes of structures with appropriate languages $\mathscr{L}_A$ and $\mathscr{L}_B$ respectively, and $I \subseteq J$. (I and J need not be sets.)

By an *interpretation* from $\mathbf{A}$ into $\mathbf{B}$ we mean a map F from $\mathscr{L}_A$ formulae to $\mathscr{L}_B$ formulae and from $\mathscr{L}_A$ valuations to $\mathscr{L}_B$ valuations such that:

(i) If ϕ is an $\mathscr{L}_A$ sentence, $F(\phi)$ is an $\mathscr{L}_B$ sentence.

(ii) If ϕ is an $\mathscr{L}_A$ formula, f a valuation on $\mathscr{L}_A$ for A_i $(i \in I)$, then: $F(f)$ is a valuation on $\mathscr{L}_B$ for B_i and $A_i \models_f \phi$ iff $B_i \models_{F(f)} F(\phi)$.

If there is an interpretation from $\mathbf{A}$ to $\mathbf{B}$ we say that $\mathbf{A}$ is *interpretable* in $\mathbf{B}$. If $\mathbf{A}$ is interpretable in $\mathbf{B}$ and $\mathbf{B}$ is interpretable in $\mathbf{A}$ then we say that $\mathbf{A}$ and $\mathbf{B}$ are *bi-interpretable*. (Note that in this case $I = J$.)

LEMMA 1. *Let* $\mathbf{A} = \{A_i : i \in I\}$ *and* $\mathbf{B} = \{B_i : i \in J\}$ *be classes of structures with appropriate languages* $\mathscr{L}_A$ *and* $\mathscr{L}_B$ *respectively, and* $I \subseteq J$.

1. *If* $\mathbf{A}$ *is interpretable in* $\mathbf{B}$ *then for all* $i, j \in I$: $B_i = \mathscr{L}_B B_j$ *implies that* $A_i \equiv \mathscr{L}_A A_j$.
2. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are bi-interpretable then for all* $i, j \in I$: $A_i \equiv \mathscr{L}_A A_j$ *iff* $B_i \equiv \mathscr{L}_B B_j$.

Proof. 1. Suppose $B_i \equiv \mathscr{L}_B B_j$. Let ϕ be an $\mathscr{L}_A$ sentence such that $A_i \models \phi$. Let F denote the interpretation from **A** to **B**, then $F(\phi)$ is an $\mathscr{L}_B$ sentence and so $B_i \models F(\phi)$. Hence $B_j \models F(\phi)$ and so $A_j \models \phi$. Thus $A_i \equiv \mathscr{L}_A A_j$.

2. Immediate from 1. □

We will use Lemma 1 to show that the elementary equivalence of a pair of structures in one class implies the elementary equivalence of a pair of structures in another class.

Suppose every formula of $\mathscr{L}_A$ is equivalent to a formula constructed from atomic formulae which belong to a certain set of atomic formulae which we will refer to as canonical formulae. To define an interpretation F from **A** to **B**, it suffices to:

(i) Define F on canonical formulae.
(ii) For formulae of the form $(\exists\square)\phi$, to define $F((\exists\square)\phi)$ in terms of $F(\phi)$.
(iii) Define F on valuations.

F then extends to a full interpretation by recursion and equivalence of formulae as follows:

If $F(\phi)$ and $F(\psi)$ have been defined put $F(\phi \wedge \psi) = F(\phi) \wedge F(\psi)$ and $F(\neg\phi) = \neg F(\phi)$. If ϕ is not constructed from canonical formulae choose a formula $\bar{\phi}$ equivalent to ϕ, such that $\bar{\phi}$ is constructed from canonical formulae, and put $F(\phi) = F(\bar{\phi})$.

LEMMA 2. *The following pairs of classes are bi-interpretable:*
(i) *R and K^R*
(ii) *L and K^L*
(iii) *M and K^M*
(iv) *S and K^S.*

Proof. (i) *Interpretation from R to K^R:*
For canonical formulae:

For $i, j, k < \omega$ define:

$$F(x_i = x_k) = (\rho_i = \rho_k)$$

$$F(x_i = 0) = \mathrm{Zero}(\rho_i)$$

$$F(x_i = 1) = \mathrm{One}(\rho_i)$$

$$F(x_i = e) = \mathrm{Id}(\rho_i)$$

$$F(x_i \circ x_j = x_k) = \mathrm{Inter}(\rho_k, \rho_i, \rho_j)$$

$F(x_i + x_j + x_k) = \mathrm{Union}(\rho_k, \rho_i, \rho_j)$

$F(x_i \circ x_j = x_k) = \mathrm{Prod}(\rho_k, \rho_i, \rho_j)$

$F(x_i = x_k') = \mathrm{Comp}(\rho_i, \rho_k)$

Put $F((\exists x_i)\phi) = (\exists \rho_i)F(\phi)$.

If f is a valuation on $\mathscr{L}_R$ for R_λ define $F(f)$ on $\mathscr{L}\mathscr{R}$ for $\langle \lambda, \mathscr{R}(\lambda)\rangle$ by:

For $i < \omega$ put $F(f)(w_i) = 0$ and put $F(f)(\rho_i) = f(x_i)$. Thus ρ_i is used to translate x_i while w_i is not used to represent any necessary information.

Interpretation from K^R to R:
Besides the usual abbreviations for $\mathscr{L}_R$ we also need:
For terms t, s, r define:

$\mathrm{Point}(t) : (0 \prec t) \wedge (t \leq e)$

In R_λ this says that t is of the form $\{(a, a)\}$ and hence specifies an element of λ viz. a.

$\mathrm{Pair}(t, s, r) : (0 \prec t) \wedge (s \circ t \circ r = t)$

In R_λ if $s = \{(a, a)\}$ and $r = \{(b, b)\}$ then this says that $t = \{(a, b)\}$.

$\mathrm{Rel}(t, s, r) : (\exists x)(\mathrm{pair}(x, s, r) \wedge x \leq t)$

In R_λ if $s = \{(a, a)\}$ and $r = \{(b, b)\}$ then this says that $(a, b) \in t$.

Now for $i, j, k < \omega$ define:

$F(w_i = w_k) = (x_{2i} = x_{2k})$

$F(w_i \rho_j w_k) = \mathrm{Rel}(x_{2j+1}, x_{2i}, x_{2k})$

Put $F((\exists w_i)\phi) = (\exists x_{2i})(\mathrm{point}(x_{2i}) \wedge F(\phi))$ and put $F((\exists \rho_i)\phi) = (\exists x_{2i+1})F(\phi)$.

If f is a valuation on $\mathscr{L}\mathscr{R}$ for $\langle \lambda, \mathscr{R}(\lambda)\rangle$ define $F(f)$ on $\mathscr{L}_R$ for R_λ by:
For $i < \omega$ put $F(f)(x_{2i}) = \{(f(w_i), f(w_i))\}$ and put $F(f)(x_{2i+1}) = f(\rho_i)$. Hence elements of λ are represented by relations of the form $\{(a, a)\}$. For $i < \omega x_{2i}$ is used to translate w_i and x_{2i+1} is used to translate ρ_i. $\square$

(ii) *Interpretation from L to K^L*
For canonical formulae:

For $i, j, k < \omega$ put:

$F(x_i = x_k) = (\rho_i = \rho_k)$

$F(x_i \cdot x_j = x_k) = \mathrm{Inter}(\rho_k, \rho_i, \rho_j)$

$F(x_i + x_j = x_k) = \mathrm{Join}(\rho_k, \rho_i, \rho_j)$

Put $F((\exists x_i)\phi) = (\exists \rho_i)F(\phi)$

If f is a valuation on $\mathscr{L}_L$ for L_λ define $F(f)$ on $\mathscr{L}\mathscr{R}$ for $\langle \lambda, \mathrm{Part}(\lambda)\rangle$ as follows:

For $i < \omega$ put $F(f)(w_i) = 0$ and put $F(f)(\rho_i) = f(x_i)$. Thus ρ_i is used to translate x_i. (w_i is not used to represent any necessary information.)

Interpretation from K^L to L
Besides the usual abbreviations for $\mathscr{L}_L$ we also need:

For terms t, s, r define:

$\mathrm{Zero}(t) : (\forall x)(t \leq x)$

i.e. 't is the smallest element'.

$\mathrm{One}(t) : (\forall x)(x \leq t)$

i.e. 't is the largest element'.

$\mathrm{Atom}(t) : (\exists x)(\mathrm{Zero}(x) \wedge (x \prec t))$

i.e. 't is an atom'.

$\text{Co-atom}(t) : (\exists x)(\mathrm{One}(x) \wedge (t \prec x))$

i.e. 't is a co-atom'.

$\mathrm{Links}(s, t) : (\exists x)(\mathrm{Atom}\ (x) \wedge x \neq s \wedge x \neq t \wedge x \leq s + t).$

If s, t are atoms in L_λ this says that the intersection of their unique two element blocks is non-empty.

$$\mathrm{Point}(t) : \text{co-atom}(t) \wedge (\forall y)(\forall z)(\mathrm{Atom}(y) \wedge \mathrm{Atom}(z)$$
$$\wedge (y \nleq t) \wedge (z \nleq t)) \Rightarrow \mathrm{Link}(y, z))$$

In L_λ this says that t is a co-atom and hence has two blocks, also any two atoms not below t have their two-element blocks intersecting. Hence at least one block of t is trivial. If $\lambda \geq 3$ then the other block of t is non-trivial and so t specifies a unique element of λ viz. the element in the trivial block.

$\mathrm{Comp}(t, s) : \mathrm{One}(t + s) \wedge \mathrm{Zero}(t \cdot s)$

i.e. 't and s are complements'.

$\mathrm{Atom}(t, s, r) : \mathrm{Atom}(t) \wedge (s \neq r) \wedge \mathrm{Comp}(t, r) \wedge \mathrm{Comp}(t, s)$

In L_λ where $\lambda \geq 3$, if s and r specify elements of λ then this says that these elements are distinct and that t is the atom whose two-element block contains these elements.

$$\mathrm{Equ}(t, s, r) : (s = r) \wedge (\exists x)(\mathrm{Atom}(x, r, s) \wedge (x \leq t))$$

In L_λ where $\lambda \geq 3$, if s and r specify elements of λ then this says that either $s = r$, or $s \neq r$ and the atom whose two-element block consists of the elements specified by s and r, is below t. Hence the elements specified by s and r are in the same block of t.

Now for $i, j, k < \omega$ define:

$F(w_i = w_k) = (x_{2i} = x_{2k})$

$F(w_i \rho_j w_k) = \mathrm{Equ}(x_{2j+1}, x_{2i}, x_{2k})$

Put $F((\exists w_i)\phi) = (\exists x_{2i})(\mathrm{point}(x_{2i}) \wedge F(\phi))$ and put $F((\exists \rho_i)\phi) = (\exists x_{2i+1})F(\phi)$.

For each valuation f on $\mathscr{L}\mathscr{R}$ for $\langle \lambda, \mathrm{Part}(\lambda) \rangle$, $\lambda \geq 3$, define $F(f)$ on $\mathscr{L}_L$ for L_λ as follows:

For $i < \omega$ put $F(f)(x_{2i})$ = the partition of λ with two blocks, one of which is $\{f(w_i)\}$ and put $F(f)(x_{2i+1}) = f(\rho_i)$. Thus elements of λ are represented by partitions with two blocks, one of which is a singleton. For $i < \omega$ x_i is used to translate w_i and x_{2i+1} is used to translate ρ_i. ☐

(iii) *Interpretation from M to K^M*

For canonical formulae:

for $i, j, k < \omega$ put:

$F(x_i = x_k) = (\rho_i = \rho_k)$

$F(x_i = e) = \mathrm{Id}(\rho_i)$

$F(x_i \circ x_j = x_k) = \mathrm{Prod}(\rho_k, \rho_i, \rho_j)$

Put $F((\exists x_i)\phi) = (\exists \rho_i)F(\phi)$

For each valuation f on $\mathscr{L}_M$ for M_λ define $F(f)$ on $\mathscr{L}\mathscr{R}$ for $\langle \lambda, \mathscr{F}(\lambda) \rangle$ as follows:

For $i < \omega$ put $F(f)(w_i) = 0$ and $F(f)(\rho_i) = f(x_i)$. (Thus for $i < \omega$ ρ_i is used to translate x_i.)

Interpretation from K^M to M

We need the following abbreviation:

For t a term define:

$$\mathrm{const}(t) : (\forall x)(t \circ x = t)$$

In M this says that t is a constant function. (If t is not constant we can find $a, b \in \lambda$ such that $t(a) \neq t(b)$. Choosing x such that $x(a) = x(b)$ we get $t \circ x \neq t$.)

Now for $i, j, k < \omega$ define:

$$F(w_i = w_k) = (x_{2i} = x_{2k})$$

$$F(w_i \rho_j w_k) = (x_{2j+1} \circ x_{2i} = x_{2k})$$

Put $F(\exists w_i)\phi) = (\exists x_{2i})(\text{const}(x_{2i}) \wedge F(\phi))$.

Put $F((\exists \rho_i)\phi) = (\exists x_{2i+1})F(\phi)$.

For each valuation f on $\mathscr{L}\mathscr{R}$ for $\langle \lambda, \mathscr{F}(\lambda) \rangle$ define $F(f)$ on $\mathscr{L}_M$ for M as follows:

For $i < \omega$ put $F(f)(x_{2i})$ = the constant function on λ with value $f(w_i)$ and put $F(f)(x_{2i+1}) = f(\rho_i)$. Thus elements of λ are represented by constant functions. For $i < \omega$ x_{2i} is used to translate w_i and x_{2i+1} is used to translate ρ_i. □

(iv) This follows from the fact that for models of the form $\langle \lambda, \text{Perm}(\lambda) \rangle$, $\mathscr{L}\mathscr{R}$ is equivalent to McKenzie's language $\mathscr{L}\mathscr{A}$. (see [7]) □

LEMMA 3. *Let $\lambda, k > 0$. The following are equivalent:*
(i) $\lambda \equiv \mathscr{L}_2 \kappa$
(ii) $\langle \lambda, \mathscr{R}(\lambda) \rangle \equiv \mathscr{L}\mathscr{R} \langle \kappa, \mathscr{R}(\kappa) \rangle$
(iii) $\langle \lambda, \text{Part}(\lambda) \rangle \equiv \mathscr{L}\mathscr{R} \langle \kappa, \text{Part}(\kappa) \rangle$
(iv) $\langle \lambda, \mathscr{F}(\lambda) \rangle \equiv \mathscr{L}\mathscr{R} \langle \kappa, \mathscr{F}(\kappa) \rangle$

Proof. For λ, κ finite the result is trivial. For infinite λ, clearly (i) $\Rightarrow$ (ii), (iv). Since partitions are definable in $\mathscr{L}\mathscr{R}$ (ii) $\Rightarrow$ (iii). Restating Rabin's result mentioned on page 160 of [9] (iii) $\Leftrightarrow$ (i). By Theorem 10 of [7] and Lemma 3 of [10] we also have (iv) $\Rightarrow$ (i). □

LEMMA 4. *Let $\lambda, \kappa > 0$. Then:*

$\langle \lambda, \mathscr{F}(\lambda) \rangle \equiv \mathscr{L}\mathscr{R} \langle \kappa, \mathscr{F}(\kappa) \rangle$ *implies* $\langle \lambda, \text{Perm}(\lambda) \rangle \equiv \mathscr{L}\mathscr{R} \langle \kappa, \text{Perm}(\kappa) \rangle$.

Proof. Clear since permutations are definable in $\mathscr{L}\mathscr{R}$. □

THEOREM 5. *Let $\lambda, \kappa > 0$. Then the following are equivalent:*
(i) $\lambda \equiv \mathscr{L}_2 \kappa$
(ii) $R_\lambda \equiv R_\kappa$
(iii) $L_\lambda \equiv L_\kappa$
(iv) $M_\lambda \equiv M_\kappa$.

Proof. Immediate from Lemmas 1, 2, 3. □

THEOREM 6. *For* $\lambda, \kappa > 0$ *each of the following implies that* $S_\lambda \equiv S_\lambda$.

(i) $\lambda \equiv \mathscr{L}_2 \kappa$
(ii) $R_\lambda \equiv R_\kappa$
(iii) $L_\lambda \equiv L_\kappa$
(iv) $M_\lambda \equiv M_\kappa$.

Proof. Immediate from Lemma 1, 2, 4 and Theorem 5. □

Now for all ordinals α define:

$$R^\alpha = R_{\aleph_\alpha}, \qquad L^\alpha = L_{\aleph_\alpha}, \qquad M^\alpha = M_{\aleph_\alpha} \text{ and } S^\alpha = S_{\aleph_\alpha}.$$

Also if α is an ordinal let $\langle \alpha, < \rangle$ denote the ordered structure with domain α and $<$ the usual strict order on ordinals.

LEMMA 7 (McKenzie [7]). *If* α, β *are ordinals then*:

$S^\alpha \equiv S^\beta$ *implies that* $\langle \alpha, < \rangle \equiv \langle \beta, < \rangle$.

COROLLARY 8. *Each of the following implies that* $\langle \alpha, < \rangle \equiv \langle \beta, < \rangle$.

(i) $\aleph_\alpha \equiv \mathscr{L}_2 \aleph_\beta$
(ii) $R^\alpha \equiv R^\beta$
(iii) $L^\alpha \equiv L^\beta$
(iv) $M^\alpha \equiv M^\beta$
(v) $S^\alpha \equiv S^\beta$.

Proof. Immediate from Lemma 7, Theorem 5 and Theorem 6. □

COROLLARY 9. *If* $\alpha, \beta < \omega$ *each of the conditions* (i)–(v) *in Corollary* 8 *are equivalent to the condition* $\alpha = \beta$.

Proof. Clearly (i)–(v) are necessary for $\alpha = \beta$. Since for finite ordinals: $\langle \alpha, < \rangle \equiv \langle \beta, < \rangle$ iff $\alpha = \beta$, they are also sufficient. □

In particular we have rederived Ježek's result that for $\alpha, \beta < \omega$ $L^\alpha \equiv L^\beta$ iff $\alpha = \beta$ (see [5]).

Shelah derived necessary and sufficient conditions on α and β for $S^\alpha \equiv S^\beta$ (see [9]). From Shelah's result we see $\langle \alpha, < \rangle \equiv \langle \beta, < \rangle$ does not imply $S^\alpha \equiv S^\beta$. Hence none of the reverse implications in Corollary 8 hold in general.

Some other structures associated with sets

We have considered only the structures R_X, L_X, M_X and S_X which seem to be the most interesting structures naturally associated with X. However, there are numerous other such structures. We mention just a few.

(i) The Boolean algebra of all subsets of X.

$$P_X = \langle \mathscr{P}(X), \cap, \cup, ', \varnothing, X \rangle.$$

Since all infinite atomic Boolean algebras are elementary equivalent [3] we see that $P_\lambda \equiv P_\kappa$ iff λ, κ are both finite and equal or λ, κ are both infinite.

(ii) The semilattice of all order relations on X

$$O_X = \langle \mathscr{O}(X), \cap \rangle$$

where $\mathscr{O}(X)$ is the set of all order relations on X, and $\cap$ is intersection of relations.

It is not difficult to see that the class $\{O_\lambda : \lambda > 0\}$ is interpretable in the class $\{\langle \lambda, \mathscr{O}(\lambda) \rangle : \lambda > 0\}$ (for which $\mathscr{L}\mathscr{R}$ is the appropriate language), but is the converse true? This remains an open question. The problem is how to represent elements of λ by order relations in such a way that "$a < b$" can be expressed for any given order relation $<$.

(iii) The word semigroup on X.

$$W_X = \{\mathscr{W}(X), *\}$$

where $\mathscr{W}(X)$ is the set of non-empty words on X (i.e. tuples of elements of X) and $*$ represents concatenation of words.

Note that for all $\lambda > 0$ W_λ is the free semigroup with λ generators. Hence if $\kappa \leq \lambda$ W_κ is a retract of W_λ and so every positive or universal sentence holding in W_λ holds in W_κ, in notation $W_\lambda(P)W_\kappa$ and $W_\lambda(\forall)W_\kappa$. When is $W_\lambda \equiv W_\kappa$ and how is this related to our results? (More generally we may examine free algebras of any kind.) ☐

Ultrauniversality

Let **C** be an elementary class or a variety and let $A \in C$. A is called *ultrauniversal* in **C** (Bruyns and Rose [1]) iff it satisfies one, and hence all, of the following equivalent conditions:

(i) Every member of C is embeddable in an ultraproduct of A.
(ii) If ϕ is an existential sentence holding in some member of C then ϕ holds in A.
(iii) If ϕ is a universal sentence holding in A then ϕ holds in all members of C.

We know that for $\lambda \geq \aleph_0$ L_λ and S_λ are ultrauniversal in the variety of lattices and groups respectively [1]. We also have:

LEMMA 10. *If $\lambda \geq \aleph_0$ M_λ is ultrauniversal in the variety of monoids.*

Proof. Let A be a monoid. Let $\{B_i : i \in I\}$ be the collection of all finitely generated submonoids of A. Since any algebra is embeddable in an ultraproduct of its finitely generated subalgebras [2] there is an ultrafilter $\mathscr{D}$ over I such that A is embeddable in $\Pi_{\mathscr{D}} B_i$. Now every monoid N is embeddable in M_κ where $\kappa = |N|$. Also if $\kappa \leq \lambda$ then M_κ is embeddable in M_λ. Noting that B_i's are countable we have that A is embeddable in $\Pi_{\mathscr{D}} M_\lambda$ for any $\lambda \geq \aleph_0$, and so the result follows. □

Let Z denote either L, M or S.

THEOREM 11. *Let $\lambda, \kappa \geq \aleph_0$. Then Z_λ and Z_κ satisfy the same universal and existential sentences.*

Proof. Immediate since Z_λ, Z_κ are both ultrauniversal. □

An interesting problem is to find the smallest $n < \omega$ such that there are infinite cardinals λ and κ and a Π_n or Σ_n sentence ϕ which holds in Z_λ but fails in Z_κ, where $Z = L$, M, S, or R etc. Theorem 11 tells us that for $Z = L$, M or S, n is strictly greater than 1. Can this be extended to $Z = R$? □

REFERENCES

[1] BRUYNS, P. and ROSE, H, *Varieties with cofinal sets*. Preprint.
[2] BURRIS, S. and SANKAPANAVAR, H. P. *A course in Universal Algebra*, Springer-Verlag, New York, 1981, p. 213.
[3] CHANG, C. C. and KEISLER, H. J., *Model Theory*, North-Holland, Amsterdam, 1973, p. 300.
[4] GRÄTZER, G., *General Lattice Theory*, Birkhausen Verlag, Basel & Stuttgart, 1978.

[5] JEŽEK, J., *Elementary non-equivalent infinite partition lattices*, Algebra Universalis *20* (1985), 132 133.
[6] JÓNSSON, B. and TARSKI, A., *Boolean algebras with operators, II*, Amer. J. Math. *74* (1952), 127–162.
[7] MCKENZIE, R., *On elementary types of symmetric groups*, Algebra Universalis *1* (1971), 13–20.
[8] MCKINSEY, J. C. C., *Postulates for the calculus of binary relations*, J. Sym. Logic *5* (1940), 85–97.
[9] SHELAH, S., *First order theory of permutation groups*, Israel J. Math. *14* (1973), 149–162. See also Errata to the above Israel J. Math. *15* (1973), 437–141.
[10] SHELAH, S., *There are just four second order quantifiers*, Israel J. Math. *15* (1973), 282–300.

Department of Mathematics
University of Cape Town
Republic of South Africa